On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonians flows

Claude Viterbo
Centre de Mathématiques Laurent Schwartz
UMR 7640 du CNRS
École Polytechnique - 91128 Palaiseau, France
viterbo@math.polytechnique.fr

22nd November 2006

Abstract

We show that if a sequence of Hamiltonian flows has a C^0 limit, and if the generating Hamiltonians of the sequence have a limit, this limit is uniquely determined by the limiting C^0 flow. This answers a question by Y.G. Oh in [Oh04].

1 Introduction

Let $H_k(t,x)$ be a sequence of Hamiltonians with flow φ_k^t on a symplectic manifold (M,ω), such that $\lim_k H_k(t,x) = H(t,x)$ and $\lim_k \varphi_k^t = \varphi^t$ where limits are intended as C^0 limits. Can we say that φ is the flow of H ? In the case where H has a flow (e.g. H is $C^{1,1}$) this has been proved in [Vit92], and one could alternatively use the methods of [Hof90]. However if H is only C^0, it is not easy to make sense of this question, since the flow of a C^0 hamiltonian is not defined.

This question is not as artificial as the reader may think, and has apparently been asked by Y.G. Oh in the framework of C^0-Hamiltonians ([Oh04], this seems to be related to Question 3.11 or 3.20). It is sufficient to solve this question for the case where H is continuous and $\varphi = Id$. Do we necessarily have $H = 0$ in this case ?

Of course we could have $H(t,x) = h(t)$ and we exclude this case by normalizing the Hamiltonians, in the compact case by imposing the condition

$$\int_M H_k(t,x)\omega^n = 0$$

and in the non-compact case by assuming they have compact support.

Our aim in this short note is to give a positive answer to Oh's question. We denote by $C^{1,1}$ the set of differentiable functions with Lipschitz derivative.

Theorem 1.1. *Let $H_n(t,z)$ be a sequence of $C^{1,1}$ Hamiltonians on (M,ω), normalized as above, and such that H_n converges in the C^0 topology to some continuous function $H(t,z)$. Let φ_n^t be the flow of H_n. Then if φ_n^t converges to* Id *in the C^0 topology, we have $H = 0$.*

Remark 1.2. a) Throughout the paper, by C^0 convergence of φ_n^t to φ^t, we always mean C^0 convergence uniform in t, for t in $[0,1]$. In other words,

$$\forall \varepsilon > 0 \; \exists N_0 \; \forall t \in [0,1] \; \forall N > N_0, \; \|\varphi_n^t - \varphi^t\|_{C^0} \leq \varepsilon$$

b) Note that if we do not assume that H_n converges, the theorem does not hold. Indeed, consider a non-zero Hamiltonian H_0 supported in the unit ball. Then the sequence $nH_0(nz)$ does not converge, but the time one flow C^0, does converge to the identity.

c) According to Y.G. Oh, one can adapt the proof of the theorem to the case where convergence is in Hofer norm, i.e. the norm given by

$$\|H\| = \int_0^1 \left[\max_{x\in M} H(t,x) - \min_{x\in M} H(t,x) \right] dt$$

As a Corollary we get

Corollary 1.3. *Let φ_n^t, ψ_n^t be sequences of Hamiltonian flows associated to $H_n(t,x), K_n(t,x)$. Assume*

$$\lim_n H_n = H, \lim_n K_n = K$$

and

$$\lim_n \varphi_n^t = \lim_n \psi_n^t = \rho^t$$

where all limits are intended as C^0 limits. Then

$$H = K$$

Proof of Corollary, assuming the theorem. Indeed, $(\varphi_n^t)^{-1} \circ (\psi_n^t)$ C^0 converges to the identity, and is generated by $(K_n - H_n)(t, \varphi_n^t(z))$, hence C^0 converges to $(H-K)(t, \rho^t(z))$. Thus, according to the theorem we have $(H-K)(t, \rho^t(z)) = 0$ hence $H = K$.

□

We thank Albert Fathi for drawing our attention to this problem, and Y.G. Oh for raising the question and for some useful comments.

2 On C^0-limits of Lagrangian submanifolds

Consider the problem of a topological submanifold (i.e. C^0) L in T^*N, that would be a C^0 limit of C^1 Lagrangian submanifolds. According to [LS94], if L is C^1, it is necessarily Lagrangian. When L is C^0, the meaning of "Lagrangian" is unclear. However if L is a graph in the cotangent bundle, $L = \{(x, p(x)) \mid x \in N\}$, requiring that $p(x)dx$ is closed makes sense even if $p \in C^0$. Indeed we may interpretate this as meaning that $p(x)dx$ is closed in the distribution sense, as suggested by Michael Herman ([Her89] definition 8.13 page 60). In our case we wish to prove that if L is a non Lagrangian C^0 graph, we may not approximate it by Lagrangian submanifolds. Our crucial assertion is

Proposition 2.1. *Let N be a closed manifold that is the total space of an S^1 fibration. Let p be a continuous section of T^*N which, considered as a one-form, is not closed in the sense of distributions. Then, there exists $f \in C^\infty$ such that $p(x) - df(x)$ does not vanish on N.*

Corollary 2.2. *Let L be the graph of a one form p and assume that any neighbourhood U of L contains a smooth exact Lagrangian submanifold of T^*M. Then p is closed in the sense of distributions.*

In particular if L_n is a sequence of exact Lagrangians submanifolds in T^*M, L is the C^0 graph of a one form p, and L_n converges C^0 to L, then p is closed.

This corollary is proved at the end of this section.

Remark 2.3. (a). Note that once we have a C^1 function f such that $p(x) - df(x)$ does not vanish, the same holds if we replace f by any g close to f. Thus, we can replace f by a smooth approximation. In the sequel we shall thus not bother about the smoothness of the solution.

(b). Note that a smooth fibration with fiber S^1, that is a principal bundle with group $Diff(S^1)$ is equivalent to a principal bundle with fiber the group S^1, since the inclusion of S^1 into $Diff(S^1)$ induces an isomorphism of homotopy groups.

(c). Our corollary can be compared with Theorem 2 in [Sik91]. Even though it is stated there with stronger assumptions, Sikorav's proof yields our corollary when p is smooth. It can probably be adapted to the case where p is only C^0 using an analog of our lemma 2.5.

Note that in theorem 2 of [LS94] the authors consider a sequence $\varphi_n : V \longrightarrow T^*M$ of maps converging C^0 to φ_∞. They prove that if the $L_n = \varphi_n(V)$ are Lagrangians embeddings, and φ_∞ is smooth, then $\varphi_\infty(V)$ is Lagrangian, provided $\pi_2(T^*M, L_n) = 0$. In particular, up to a symplectomorphism, L_n may be assumed to be exact. In this case however, the smoothness of φ seems to be crucial, and moroever convergence here is meant in a stronger sense (convergence of embeddings rather than Hausdorff).

Our Corollary trades the exactness requirement against a weaker assumption on convergence. This exactness of L_n is crucial as can be seen from the following example.

Any submanifold can be approximated in the Hausdorff topology by a (non-exact) Lagrangian one. Indeed, given V, we may approximate it by a union of small Lagrangian tori, each being contained in a Darboux chart near V. On the union of such tori, we may perform a Polterovich surgery, in order to obtain a connected Lagrangian submanifold.

Note that quite obviously the map $H_1(V) \to H_1(N)$ is not injective in this case: each torus produces a lot of 1-cycles of V, which go to zero in N. This non-injectivity of the map $H_1(V) \to H_1(N)$ is crucial in this counterexample: if the map was injective, we could translate Ł$_n$ by a closed form to make it exact, and our Corollary would apply.

The proof of the proposition will require some lemmata.
Under the assumptions of the proposition, let V be the base of the circle fibration, and $y : N \longrightarrow V$ be the projection. For y in a domain of trivialization of the fibration, we consider coordinates (θ, y) where $\theta \in S^1$. Note that we may always assume to be given an invariant measure (by the circle action) on N. We also set $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$ where $\theta \in S^1, y \in V$, $\pi(\theta, y) \in \mathbb{R}$ and r is a section of T^*V parametrized by $\theta \in S^1$.

Lemma 2.4. *Assume for some $y \in V$ we have*

$$P(y) = \int_{S^1} \pi(\theta, y) d\theta \neq 0$$

Then there exists $f \in C^1(S^1 \times V, \mathbb{R})$ such that $p(\theta, y) - df(\theta, y)$ does not vanish. The same holds for N the total space of a circle fibration..

Proof. Let $p(\theta, y) = (\pi(\theta, y), r(\theta, y))$ be a one form on $S^1 \times V$ that is a section of $T^*(S^1 \times V)$. We look for a function $f(\theta, y)$ such that $p(\theta, y) - df(\theta, y)$ never vanishes. Let us first try to solve

$$\pi(\theta, y) - \frac{\partial f}{\partial \theta}(\theta, y) = \varepsilon(\theta, y)$$

Then this is solvable if and only if

$$\int_{S^1} (\pi(\theta, y) - \varepsilon(\theta, y)) d\theta = 0$$

and thus, denoting by $P(y) = \int_{S^1} \pi(\theta, y) d\theta$, we can choose ε non vanishing outside a neighbourhood of the set $Z = \{(\theta, y) \mid P(y) = 0\}$ (e.g. take $\varepsilon(\theta, y) = P(y)$). In general we cannot choose ε to be non-zero in such a neighbourhood.

Note that f is well-defined up to a function of y. Also, we can assume f to be smooth, provided we took care to choose $\varepsilon(\theta, y) - \pi(\theta, y)$ smooth.

Now we need to find $h \in C^1(V, \mathbb{R})$ such that $p(\theta, y) - df(\theta, y) - dh(y)$ does not vanish on Z. But if the projection of Z on V, U, is not all of V, we can find a function h on V with no critical point in U. Multiplying h by a large constant, we may assume dh to be arbitrarily large. Then $p(\theta, y) - df(\theta, y) - dh(y)$ will not vanish for all (θ, y) with $y \in U$, and thus for $(\theta, y) \in Z$.

□

Lemma 2.5. *Assume p is C^0 on N, and consider a smooth circle fibration of N. Assume for any curve γ, C^∞ close to a fiber of the fibration, we have*

$$\int_{S^1} p(\gamma(t))\dot{\gamma}(t) dt = 0$$

Then p is closed in the sense of distributions.

Proof. We shall take local coordinates (θ, y) in the neighbourhood of a fiber, and let $\eta(\theta, y)$ be a smooth vector field on N.

Let α be a continuous one-form. We wish to compute the integral of α over the curve $t \to (t, y + \varepsilon\eta(t, y))$. This will be

$$\int_{S^1} \alpha(\theta, y + \varepsilon\eta(\theta, y))(1, \varepsilon\frac{\partial}{\partial\theta}\eta(\theta, y)) d\theta$$

Writing $\alpha = \alpha_\theta d\theta + \alpha_y dy$ we can rewrite the above as

$$\int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y))\frac{\partial}{\partial\theta}\eta(\theta, y)d\theta$$

Now averaging this over $y \in V$, and differentiating with respect to ε, we get

$$\frac{\partial}{\partial\varepsilon}\int_V\int_{S^1} \alpha_\theta(\theta, y + \varepsilon\eta(\theta, y)) + \varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y))\frac{\partial}{\partial\theta}\eta(\theta, y)d\theta dy =$$
$$-\int_V\int_{S^1}\Big[\alpha_\theta(\theta, y + \varepsilon\eta(\theta, y))\nabla_y \cdot \eta(\theta, y) + \alpha_y(\theta, y + \varepsilon\eta(\theta, y))\frac{\partial}{\partial\theta}\eta(\theta, y) +$$
$$\varepsilon\alpha_y(\theta, y + \varepsilon\eta(\theta, y)) \wedge d_y\eta(\theta, y)\Big]d\theta dy$$

This is obtained by applying the change of variable $(\theta', y') = (\theta, y + \varepsilon\eta(\theta, y))$ to the

$$\int_{V\times S^1} f(\theta, y + \varepsilon\eta(\theta, y))d\theta dy$$

to get

$$\int_{V\times S^1} f(\theta, y')dy$$

where

$$dy' = dy + \varepsilon\eta(\theta, y') + o(\varepsilon)$$

so that

$$dy = dy' - \varepsilon\eta(\theta, y') + o(\varepsilon)$$

and

$$\det(dy') = \det(dy) - \varepsilon\nabla_y \cdot \eta(\theta, y)$$

remembering that trace $d\eta = \nabla \cdot \eta$

$$\int_{V\times S^1} f(\theta, y')dy = \int_{V\times S^1} f(\theta, y')dy' - \varepsilon\int_{V\times S^1} f(\theta, y')\frac{\partial}{\partial y'}\eta(\theta, y')dy'$$

and denoting by ∇_y the nabla operator with respect to the y variables, where all derivatives should be understood in the distributional sense. Now taking the above for $\varepsilon = 0$, we get

$$\int_V\int_{S^1}\left[(\alpha_\theta\nabla_y \cdot \eta)(\theta, y) + \alpha_y(\theta, y)\frac{\partial}{\partial\theta}\eta(\theta, y)\right]d\theta dy$$

Integrating by parts we get

$$\int_V \int_{S^1} \left[(\nabla_y \alpha_\theta)(\theta, y)\eta(\theta, y) - (\frac{\partial}{\partial\theta}\alpha_y)(\theta, y)\eta(\theta, y) \right] d\theta dy =$$
$$\int_V \int_{S^1} \left[(\nabla_y \alpha_\theta)(\theta, y) - \frac{\partial}{\partial\theta}\alpha_y(\theta, y) \right] \eta(\theta, y) d\theta dy$$

The last line equals the integration of $d\alpha$ against the bivector $\frac{\partial}{\partial\theta} \wedge (0, \eta)$. As this vanishes for all η, means that $\imath_{\frac{\partial}{\partial\theta}} d\alpha$ vanishes as a distribution (or current).

We thus proved that if for all η the integration of α over the loop $t \to (t, y + \varepsilon\eta(t, y))$ has vanishing derivative, then $\imath_{\frac{\partial}{\partial\theta}} d\alpha$ is identically zero. Now if we slightly modify our fibration, and apply the same argument, we get that $\imath_Z d\alpha = 0$ for any vector field Z tangent to the fiber of a circle fibration of N, close to the given one. The next lemma allows us to conclude the proof.

Lemma 2.6. *Assume α is a continuous form such that for any vector field Z, tangent to a circle fibration of N and close to Z_0, we have $i_Z d\alpha = 0$. Then $d\alpha = 0$ as a distribution.*

Proof. Indeed, it is enough to show that our assumption implies that $i_Z d\alpha$ vanishes for all vector fields Z.

First of all, the problem is local: using a partition of unity, it is enough to show that $i_Z d\alpha = 0$ holds for any Z supported in a small set, tangent to a fibration close to Z_0.

Now since Z_0 does not vanish, any vector field C^1 close to Z_0 has a flow box near z_0, hence a small diffeomorphism makes it tangent to Z_0. Thus locally, the set of Z such that $i_Z d\alpha = 0$ is open in the C^∞ topology, and thus, by considering i_{Z-Z_0}, any small vector field supported in the neighbourhood of z_0 satisfies $i_Z d\alpha = 0$. □

□

Proof of the proposition. According to the second lemma, if p is not closed, using a vector field, we may smoothly perturb the fibration, π so that one of the fibers satisfies $\int_{\pi^{-1}(y)} p \neq 0$. Then, using this new fibration and the first lemma, we see that there is a function f such that $p(x) - df(x)$ does not vanish.

□

Proof of Corollary, following [LS94]. First of all if L_n converges to L, then $L_n \times 0_{S^1} \subset T^*(N \times S^1)$ converges to $L \times 0_{S^1}$, and this will be the graph of p, considered as a one-form on $N \times S^1$. Now if p is closed on N, its extension to $N \times S^1$ is also closed, since

$$\int_{N\times S^1} \left[\frac{\partial}{\partial x_i} p_j(x) - \frac{\partial}{\partial x_j} p_i(x) \right] \varphi(x,\theta) dx d\theta$$

defined as

$$-\int_{N\times S^1} \left[p_j(x) \frac{\partial}{\partial x_i} \varphi(x,\theta) - p_i(x) \frac{\partial}{\partial x_j} \varphi(x,\theta) \right] dx d\theta$$

is equal to

$$\int_{N\times S^1} \left[p_j(x) \frac{\partial}{\partial x_i} \bar{\varphi}(x) - p_i(x) \frac{\partial}{\partial x_j} \bar{\varphi}(x) \right] dx d\theta$$

where we set $\bar{\varphi}(x) = \int_{S^1} \varphi(x,\theta) d\theta$, so that p is closed (in the sense of distributions) as a one form on N if and only if it is closed (in the sense of distributions) as a one form on $N \times S^1$.

According to the above lemma, we see that we may, using a Hamiltonian symplectomorphism, send L away from the zero section (by $(x,p) \to (x, p - df(x))$) and thus any Lagrangian submanifold L_n in a neighbourhood of L will also be sent to $T^*N \setminus 0_N$ and thus may be disjoined from itself by a small Hamiltonian isotopy, since $(x,p) \to (x, \lambda p)$ is conformal, and thus induces a Hamiltonian isotopy on exact Lagrangians. But this is impossible according to Gromov's theorem ([Gro85] p. 330). $\square$

3 Proof of the theorem

Lemma 3.1. *Let $K(t,z)$ be a Hamiltonian in T^*N, with flow ψ^t. Then the embedding*

$$\begin{aligned} \Psi : [0,1] \times N &\longrightarrow T^*([0,1] \times N) \\ (t,z) &\longrightarrow (t, -K(t, \psi^t(z)), \psi^t(z)) \end{aligned}$$

is exact Lagrangian. If moreover $\psi^t(z)$ and $K(t,z)$ are 1-periodic in t, the Lagrangian submanifold of $T^*(S^1 \times N)$ thus obtained is also exact, after maybe changing K by a constant.

Proof. Indeed, if λ is the Liouville form, and denoting by d the differential with respect to x, while D is the differential with respect to both t and x,

$$\Psi^*(\lambda + h dt) = (\psi^t)^*(\lambda) + (\lambda(\frac{d}{dt}\psi^t(z))) - K(t, \psi^t(z)) dt$$

$$= (\psi^t)^*(\lambda) + (\psi^t)^*[i_{X_K}\lambda - K(t,z)]dt$$

Since

$$\frac{d}{ds}(\psi^s)^*(\lambda) = (\psi^s)^*(L_{X_K}\lambda)) = (\psi^s)^*(di_{X_K}\lambda + i_{X_K}d\lambda) = d[(\psi^s)^*(i_{X_K}\lambda + K)]ds$$

we have

$$(\psi^t)^*\lambda - \lambda = d\int_0^t [(\psi^s)^*(i_{X_K}\lambda + K)]ds = dF(t,x)$$

and thus

$$\Psi^*(\lambda + hdt) = dF(t,z) + \frac{\partial}{\partial t}F(t,z)dt = DF(t,z)$$

Note that if K is 1 periodic in time, and $\psi^1 = \psi^0 = \mathrm{Id}$ (this implies $\psi^{t+1} = \psi^t$)we get a Lagrangian submanifold in $T^*(S^1 \times N)$. This Lagrangian will be exact, provided we change K by some constant, by the following arguments:

(a). $F(t+1,z) - F(t,z)$ is constant in time, since

$$\frac{d}{dt}(F(t+1,z) - F(t,z)) = (\psi^{t+1})^*(i_{X_K}\lambda + K) - (\psi^t)^*(i_{X_K}\lambda + K)$$

(b). $F(1,z) - F(0,z)$ is constant in z, since

$$dF(t+1,z) - dF(t,z) = (\psi^{t+1})^*(\lambda) - (\psi^t)^*(\lambda)$$

(c). According to (a) and (b), $F(t+1,z) - F(t,z)$ is constant c. Since changing K by a constant c, changes $F(1,z) - F(0,z)$ by c. The proof is now clear.

$\square$

Now let $H(t,z)$ be a Hamiltonian on (M,ω). We associate to it the Lagrangian manifold

(1) $$\Lambda_H = \{(t, -H(t,\varphi^t(x)), x, \varphi^t(x))\}$$

By a simple computation as above, it is indeed Lagrangian. If φ^t is C^0 close to the identity, by Weinstein neighbourhood's theorem Λ_H will be contained in $T^*([0,1] \times \Delta_M)$ where Δ_M is the diagonal in $M \times \overline{M}$. As a submanifold of $T^*([0,1] \times \Delta_M)$ it will then be exact, since it is constructed as the image of the above map Ψ associated to $K(t,x_1,x_2) = H(t,x_2)$.

Moreover, according to the lemma, if near $t = 0$ and $t = 1$ we have both that H vanishes and that $\varphi^t(z) = z$, Λ may be closed to a Lagrangian submanifold of $T^*(S^1) \times T^*N$ and , after shifting H by a constant, this Lagrangian will also be exact.

This being said, to a C^1 map $\chi : [0,1] \to [0,1]$ and a Hamiltonian $H(t,z)$ with flow φ^t we associate the flow

$$\varphi_\chi^t(z) = \varphi^{\chi(t)}(z)$$

generated by

$$H_\chi(t,z) = \chi'(t)H(\chi(t),z)$$

Now if $H_n(t,z)$ is a sequence converging to $H(t,z)$ such that φ_n^t converges to the identity map, the sequence $H_{n,\chi}(t,z)$ converges to $H_\chi(t,z)$, and $\varphi_{n,\chi}^t$ converges to identity.

We shall assume χ is identically zero in a neighbourhood of 0 and 1.

Consider now the Lagrangians

$$\Lambda_n = \{t, -H_{n,\chi}(t, \varphi_{n,\chi}^t(z)), z, \varphi_{n,\chi}^t(z)) \mid t \in [0,1], z \in M\}$$

Note that since $H_\chi(0,z) = 0$, these are Lagrangians of the type (1). Since

$$H_{n,\chi}(t, \varphi_{n,\chi}^t(z)) = \chi'(t)H_n(\chi(t),z) = 0$$

and

$$\varphi_{n,\chi}^t(z)) = \varphi_n^{\chi(t)}(z) = z$$

we may close Λ_n to an exact Lagrangian submanifold in $T^*(S^1) \times M \times \overline{M}$.

The Λ_n converge in the C^0 topology to

$$\Lambda = \{t, -H_\chi(t,z), z, z) \mid t \in [0,1], z \in M\} \subset T^*(S^1) \times M \times \overline{M}$$

Since Λ is contained in $T^*(S^1) \times \Delta_M$, where Δ_M is the diagonal in$M \times \overline{M}$ we may assume using Weinstein's theorem, that Λ_n is contained in a set symplectomorphic to a neighbourhood of the zero section in $T^*(S^1) \times \Delta_M$.

Since the Λ_n are exact Lagrangians, Λ must be Lagrangian according to Corollary 2.2. Thus there is a constant c_χ such that the form $(H_\chi(t,z) - c_\chi)dt$ must be closed in the sense of distributions. This implies that $H_\chi(t,z) = h_\chi(t)$ but since for each t the average of H_χ over M is zero, we must have

$$\chi'(t)H(\chi(t),z) = H_\chi(t,z) = c_\chi$$

for all χ satisfying the above assumption. Since H_χ vanishes at $t = 0$, we must have $c_\chi = 0$. Now it is not hard, for any $t_0 \in]0,1[$ to find such a χ with $\chi(t_0) = t_0$ and $\chi'(t_0) \neq 0$. This implies that $H(t_0,z) = 0$. Since this holds on $]0,1[$, and H is continuous, it must hold everywhere.

Remark 3.2. One would like to know whether proposition 2.1 still holds for N a general compact manifold. This does not seem to follow literally from [LS94], even though their method may be useful.

Remark 3.3. We could have also used the ideas from [Sik91] for most of our proof. We think however that proposition 2.1 is of independent interest.

References

[Gro85] Misha Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[Her89] M. R. Herman. Inégalités "à priori"pour des tores lagrangiens invariants. *Publ. Math. I.H.E.S.*, 70:47–101, 1989.

[Hof90] Helmut Hofer. On the topological properties of symplectic maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):25–38, 1990.

[LS94] François Laudenbach and Jean-Claude Sikorav. Hamiltonian disjunction and limits of lagrangian submanifolds. *Intern. Math. Res. Notes*, 4:161–168, 1994.

[Oh04] Y. G. Oh. The group of hamiltonian homeomorphisms and c^0 symplectic topology. ArXiv:math.SG/0402210, v2, August 2005.

[Sik91] Jean-Claude Sikorav. Quelques propriétés des plongements lagrangiens. *Mém. S.M.F. n°46, suppl. au Bull; S.M.F. 119 (1991).*, pages 151–167, 1991.

[Vit92] C. Viterbo. Symplectic topology as the geometry of generating functions. *Math. Annalen*, 292:685–710, 1992.